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The relationship between the symmetries of and the existence of conserved vectors for the equation $\ddot{r}+f(r)L+g(r)r=0$

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Abstract. The existence of explicit expressions for conserved vectors for the charge-monopole problem and the Kepler problem is well known. The Lie algebras of the point transformations under which the equations of motion are invariant have been found more recently. The recent discovery of an explicit expression for a Laplace-Runge-Lenz-like vector for the equation of motion $\ddot{r}+h'L/r+(hh'+kr^{-2})\hat{r}=O$, $h=h(r)$, from which one of the equations for the orbit is easily obtained, has prompted the question of what is the Lie algebra of the point symmetries of the equation $\ddot{r}+f(r)L+g(r)\hat{r}=O$ of which each of the above problems is a member. In the case that $f(r)\equiv 0$ it is well known that Laplace-Runge-Lenz-like vectors exist. The existence of such conserved vectors does not imply a particular algebraic structure of the Lie point symmetries of the equation of motion. However, the existence of such symmetries provides a systematic method for constructing the vectors.

1. Introduction

Some years ago Fradkin (1967) demonstrated that all central potential problems possessed the dynamical symmetries O_4 and SU_3. The former followed from the Lie algebra so(4) (for bounded motion, so(3, 1) for unbounded motion) of the components of two conserved vectors under the operation of taking the Poisson bracket. From the nature of the potential it is evident that the angular momentum L ($:=r\times\dot{r}$, throughout this paper the mass has been scaled to 1) is a conserved vector. The other conserved vector was a conserved vector in the plane of the motion which could be written as a linear combination of generalisations of the Laplace-Runge-Lenz vector (Laplace 1799, Runge 1923, Lenz 1924) and Hamilton's vector (Hamilton 1847) of the Kepler problem. The latter group was obtained by using the angular momentum and by constructing an appropriate generalisation of the Jauch-Hill-Fradkin tensor (Jauch and Hill 1940, Fradkin 1965) of the simple harmonic oscillator problem.

In this paper we are concerned with the connection of the symmetries of the differential equation governing the motion and the existence of conserved vectors rather than the algebraic structure of the first integrals. To this purpose the central result of Fradkin (1967) is that every central potential problem has a vector of Laplace-Runge-Lenz type in addition to the angular momentum. Peres (1979) was led to a restricted form of this result since he assumed a particular structure for the Laplace-Runge-Lenz-like vector. Yoshida (1987) demonstrated the relationship of Peres' result to that of Fradkin and later extended the existence of such a vector for all motions in the plane (Yoshida 1989).

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Generally speaking, multidimensional nonlinear dynamical systems are not integrable, are lacking in symmetry and are given to chaotic evolution. However, there are some completely integrable nonlinear systems which have become well worn paradigms. The most noteworthy of these is the Kepler problem with equation of motion (in reduced co-ordinates)

$$\ddot{\boldsymbol{r}} = -\frac{\mu \hat{\boldsymbol{r}}}{r^2}. \tag{1.1}$$

(Throughout this paper overdot means differentiation with respect to time and caret denotes a unit vector.)

In addition to the conserved angular momentum vector $\boldsymbol{L}$ and energy, the Kepler problem has two additional related conserved vectors, Hamilton's vector (Hamilton 1847)

$$\boldsymbol{K} = \dot{\boldsymbol{r}} - \frac{\mu \hat{\boldsymbol{\theta}}}{L} \tag{1.2}$$

and the better known Laplace-Runge-Lenz vector (Laplace 1799, Runge 1923, Lenz 1924)

$$\boldsymbol{J} = \boldsymbol{L} \times \boldsymbol{K} = \boldsymbol{L} \times \dot{\boldsymbol{r}} + \mu \hat{\boldsymbol{r}}. \tag{1.3}$$

The latter vector may be used to obtain the orbit equation since, if the polar angle θ is measured from $\boldsymbol{J}$, the scalar product $\boldsymbol{J} \cdot \boldsymbol{r}$ leads to

$$r = \frac{L^2}{\mu - J \cos \theta}. \tag{1.4}$$

(Hamilton's vector could equally be used to obtain the orbit equation, but the derivation is not as elegant.)

The Laplace-Runge-Lenz vector has been termed inessential (Kaplan 1986) since the conservation of angular momentum and energy implies integrability by Liouville's theorem (Whittaker 1944). This is certainly a valid and correct viewpoint in so far as integrability is concerned. However, the Laplace-Runge-Lenz vector classically leads to the orbit equation (1.4) in a trivial way and quantally provides the source of the hidden degeneracy in the spectrum of the hydrogen atom. Considerations of this nature cause us to be rather more excited than Kaplan about the existence of explicit expressions for vectors such as the Laplace-Runge-Lenz vector.

Another system which has been known for a long time to possess a conserved vector is the charge-monopole problem with equation of motion

$$\ddot{\boldsymbol{r}} = -\frac{\lambda \boldsymbol{r} \times \dot{\boldsymbol{r}}}{r^3} = -\frac{\lambda \boldsymbol{L}}{r^3} \tag{1.5}$$

which has the conserved vector (Poincaré 1896)

$$\boldsymbol{Q} = \boldsymbol{L} - \lambda \hat{\boldsymbol{r}}. \tag{1.6}$$

It is usual to regard (1.6) as a generalised angular momentum (Moreira *et al* 1985) since the surface on which the motion takes place can be obtained from (1.6) by taking the scalar product of $\boldsymbol{J}$ with $\boldsymbol{r}$. As the motion is three dimensional, this is one of the two equations required to specify the orbit. The angle θ between $\boldsymbol{J}$ and $\boldsymbol{r}$ is constant, so that the orbit lies on a right circular cone with its vertex at the origin and the direction of its axis of symmetry is $\hat{\boldsymbol{P}}$.

Although Fradkin (1965) and Yoshida (1989) have demonstrated the formal existence of conserved vectors of Laplace-Runge-Lenz type for any planar motion, the number of examples for which explicit expressions were known was limited. However, in the last decade the number of dynamical systems possessing explicit conserved vectors from which the orbit equation may be derived has been increased. Katzin and Levine (1983) and Leach (1985) derived the Laplace-Runge-Lenz-type vector

$$\boldsymbol{J} = \boldsymbol{L} \times (u\dot{\boldsymbol{r}} - \dot{u}\boldsymbol{r}) + \mu\hat{\boldsymbol{r}} \tag{1.7}$$

for the time-dependent Kepler problem with equation of motion

$$\ddot{\boldsymbol{r}} = -\ddot{\mu}\boldsymbol{r}/\mu - \mu\hat{\boldsymbol{r}}/(r^2 u) \tag{1.8}$$

where $u(t)$ is an arbitrary function of time. Then Jezewski and Mittleman (Mittleman and Jezewski 1982, Jezewski and Mittleman 1983) obtained the vector

$$\boldsymbol{J} = \frac{\boldsymbol{L} \times \dot{\boldsymbol{r}}}{L^2} + \mu z'(\theta)\hat{\boldsymbol{\theta}} + \mu z(\theta)\hat{\boldsymbol{r}} \tag{1.9}$$

where $z(\theta) = \int_{\theta_0}^{\theta} \sin(\theta - \eta)(h - \alpha\eta)^{-2}\,\mathrm{d}\eta$ and h and α are constants, for the Kepler problem with a drag law proposed by Danby (1962) which has the equation of motion

$$\ddot{\boldsymbol{r}} + \frac{\alpha\dot{\boldsymbol{r}}}{r^2} + \frac{\mu\boldsymbol{r}}{r^3} = \boldsymbol{O}. \tag{1.10}$$

(The notation used follows the usage of Gorringe and Leach (1988a) rather than that of Jezewski and Mittleman.) Recently Thompson (1987), in an investigation of the Kepler-charge monopole problem with equation of motion

$$\ddot{\boldsymbol{r}} - \frac{\boldsymbol{L}}{r^3} + \left(-\frac{1}{r^3} + \frac{k}{r^2}\right)\hat{\boldsymbol{r}} = \boldsymbol{O} \tag{1.11}$$

discovered the conserved vector

$$\boldsymbol{J} = \boldsymbol{L} \times \dot{\boldsymbol{r}} + \frac{\boldsymbol{L}}{r} + k\hat{\boldsymbol{r}}. \tag{1.12}$$

(One should note that this motion is not planar and so does not fall within the ambit of the results of Fradkin (1965) and Yoshida (1989).) All of these studies were computationally rather complicated. However, adapting a particularly simple and elegant method promoted by Collinson (1973), Leach (1987), Leach and Gorringe (1987) and Gorringe and Leach (1987, 1988a, b, 1989a, b) were not only able to recover the examples cited above but also to provide whole new classes of problems for which a Laplace-Runge-Lenz-type vector existed in explicit form. Thereby they opened up the way for a direct calculation of the orbit for these problems. We mention just the one of these which is relevant to the purpose of this paper. The equation of motion

$$\ddot{\boldsymbol{r}} + \frac{h'(r)}{r}\boldsymbol{L} + \left(hh' + \frac{k}{r^2}\right)\hat{\boldsymbol{r}} = \boldsymbol{O} \tag{1.13}$$

has the conserved Laplace-Runge-Lenz-type vector

$$\boldsymbol{J} = \boldsymbol{L} \times \dot{\boldsymbol{r}} + h\boldsymbol{L} + k\hat{\boldsymbol{r}} \tag{1.14}$$

where $h(r)$ is an arbitrary differentiable function of r (Leach and Gorringe 1988).

The richness of the Kepler problem and of the charge-monopole problem in terms of explicit expressions for conserved quantities attracted attention to the Lie point symmetries of their equations of motion. For the Kepler problem with equation of motion (1.1) the following symmetry generators were found (Leach 1981, Prince and Eliezer 1981):

$$G_1=\frac{\partial}{\partial t}\qquad G_2=t\frac{\partial}{\partial t}+\frac{2}{3}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)$$
$$G_3=z\frac{\partial}{\partial y}-y\frac{\partial}{\partial z}\qquad G_4=x\frac{\partial}{\partial z}-z\frac{\partial}{\partial x}\qquad G_5=y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}. \tag{1.15}$$

The non-zero commutation relations are

$$[G_1, G_2]=G_1\qquad [G_3, G_4]=G_5$$
$$[G_4, G_5]=G_3\qquad [G_5, G_3]=G_4 \tag{1.16}$$

from which it is evident that the Lie algebra is the direct sum $a_2\oplus so(3)$. In passing we note that G_3, G_4 and G_5 follow from the invariance under rotation of (1.1), G_1 from its invariance under time translation and G_2, the generator specifically related to the Laplace-Runge-Lenz vector, indicates invariance under the similarity transformation $(t, \boldsymbol{r})\rightarrow(\bar{t}, \bar{\boldsymbol{r}}: t=\alpha\bar{t}, \boldsymbol{r}=\alpha^{2/3}\bar{\boldsymbol{r}})$. The two-dimensional algebra a_2 of G_1 and G_2 has recently found application in the analysis of one-dimensional nonlinear second-order differential equations (Leach *et al* 1988).

The algebraic structure of the Kepler problem extends to the time-dependent Kepler problem (1.8) since they are related by a point transformation (Leach 1985).

For the charge-monopole problem (1.5) Moreira *et al* (1985) found the generators

$$G_1=\frac{\partial}{\partial t}\qquad G_2=t\frac{\partial}{\partial t}+\frac{1}{2}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)$$
$$G_3=t\left(t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)\qquad G_4=z\frac{\partial}{\partial y}-y\frac{\partial}{\partial z} \tag{1.17}$$
$$G_5=x\frac{\partial}{\partial z}-z\frac{\partial}{\partial x}\qquad G_6=y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}$$

with non-zero commutation relations

$$[G_1, G_2]=G_1\qquad [G_2, G_3]=G_3\qquad [G_3, G_1]=-2G_2$$
$$[G_4, G_5]=G_6\qquad [G_5, G_6]=G_4\qquad [G_6, G_4]=G_5. \tag{1.18}$$

This algebra is the direct sum of the subalgebras of G_1, G_2 and G_3 and G_4, G_5 and G_6 and is $sl(2, R)\oplus so(3)$. (Moreira *et al* (1985) use $so(2, 1)$ rather than $sl(2, R)$.)

We observe that the charge-monopole problem has a richer algebraic structure than the Kepler problem. This means that, as Thompson (1987) observed, they cannot be related by a point transformation.

Equation (1.13) with h constant and k non-zero is just the Kepler problem (1.1). With k zero and h non-constant it cannot be reduced to the charge-monopole problem (1.5). Nevertheless (1.13) does possess a Laplace-Runge-Lenz-type vector just as (1.1) and (1.5) do and it is of interest to determine the Lie algebra associated with it. However, to make the results of this paper of greater generality we consider the equation

$$\ddot{\boldsymbol{r}}+f(r)\boldsymbol{L}+g(r)\boldsymbol{r}=\boldsymbol{O} \tag{1.19}$$

of which (1.1), (1.5) and (1.13) are particular instances.

2. The determining equations

A second-order differential equation is invariant under a Lie point transformation generated by a symmetry generator, G, if the action of the twice extended generator, $G^{(2)}$, on the differential equation is zero whenever the differential equation holds. We shall perform the analysis in a cartesian basis as it is easier to detect computational errors due to the structure of the equations obtained. We define G to be

$$G=\tau\frac{\partial}{\partial t}+\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}+\zeta\frac{\partial}{\partial z} \tag{2.1}$$

where τ, ξ, η and ζ are functions of t, x, y and z. The twice extended operator is

$$\begin{aligned}G^{(2)}=G&+(\dot{\xi}-\dot{x}\dot{\tau})\frac{\partial}{\partial\dot{x}}+(\dot{\eta}-\dot{y}\dot{\tau})\frac{\partial}{\partial\dot{y}}+(\dot{\zeta}-\dot{z}\dot{\tau})\frac{\partial}{\partial\dot{z}}\\&+(\ddot{\xi}-2\ddot{x}\dot{\tau}-\dot{x}\ddot{\tau})\frac{\partial}{\partial\ddot{x}}+(\ddot{\eta}-2\ddot{y}\dot{\tau}-\dot{y}\ddot{\tau})\frac{\partial}{\partial\ddot{y}}+(\ddot{\zeta}-2\ddot{z}\dot{\tau}-\dot{z}\ddot{\tau})\frac{\partial}{\partial\ddot{z}}\end{aligned} \tag{2.2}$$

where

$$\dot{\tau}=\frac{\partial\tau}{\partial t}+\dot{x}\frac{\partial\tau}{\partial x}+\dot{y}\frac{\partial\tau}{\partial y}+\dot{z}\frac{\partial\tau}{\partial z} \tag{2.3}$$

$$\begin{aligned}\ddot{\tau}=&\frac{\partial^2\tau}{\partial t^2}+2\dot{x}\frac{\partial^2\tau}{\partial x\,\partial t}+2\dot{y}\frac{\partial^2\tau}{\partial y\,\partial t}+2\dot{z}\frac{\partial^2\tau}{\partial z\,\partial t}+\dot{x}^2\frac{\partial^2\tau}{\partial x^2}+\dot{y}^2\frac{\partial^2\tau}{\partial y^2}+\dot{z}^2\frac{\partial^2\tau}{\partial z^2}+2\dot{x}\dot{y}\frac{\partial^2\tau}{\partial x\,\partial y}\\&+2\dot{y}\dot{z}\frac{\partial^2\tau}{\partial y\,\partial z}+2\dot{z}\dot{x}\frac{\partial^2\tau}{\partial z\,\partial x}+\ddot{x}\frac{\partial\tau}{\partial x}+\ddot{y}\frac{\partial\tau}{\partial y}+\ddot{z}\frac{\partial\tau}{\partial z}\end{aligned} \tag{2.4}$$

etc. The three components of (1.19) are

$$\begin{aligned}&\ddot{x}+f(y\dot{z}-z\dot{y})+gx=0 \qquad \ddot{y}+f(z\dot{x}-x\dot{z})+gy=0\\&\ddot{z}+f(x\dot{y}-y\dot{x})+gz=0.\end{aligned} \tag{2.5}$$

The partial differential equations which determine τ, ξ, η and ζ are obtained thus. $G^{(2)}$, (2.2), is applied to each of (2.5) with the derivatives expanded as in (2.3) and (2.4). All second derivatives are removed using (2.5). Since we have assumed a point transformation, the terms can be collected as coefficients of various combinations of powers of the total time derivatives $\dot{x}$, $\dot{y}$ and $\dot{z}$ and coefficients of linearly independent combinations then set to zero. This process is somewhat tedious to do by hand, but it was readily handled by REDUCE. Forty-eight partial differential equations were obtained, twelve of which were superfluous being a twofold repetition of the six second-order equations for τ (equation (2.6) below). We list the equations in groups according to the order in which they were subsequently analysed:

$$\begin{aligned}&\frac{\partial^2\tau}{\partial x^2}=0 \qquad \frac{\partial^2\tau}{\partial x\,\partial y}=0 \qquad \frac{\partial^2\tau}{\partial y^2}=0\\&\frac{\partial^2\tau}{\partial y\,\partial z}=0 \qquad \frac{\partial^2\tau}{\partial z^2}=0 \qquad \frac{\partial^2\tau}{\partial z\,\partial x}=0\end{aligned} \tag{2.6}$$

$$\frac{\partial^2\xi}{\partial x^2}=2\frac{\partial^2\tau}{\partial x\,\partial t}+f\left(y\frac{\partial\tau}{\partial z}-z\frac{\partial\tau}{\partial y}\right) \qquad \frac{\partial^2\xi}{\partial y^2}=fz\frac{\partial\tau}{\partial y}$$

$$\frac{\partial^2\xi}{\partial z^2}=-fy\frac{\partial\tau}{\partial z}\qquad 2\frac{\partial^2\xi}{\partial x\,\partial y}=2\frac{\partial^2\tau}{\partial y\,\partial t}+f\left(2z\frac{\partial\tau}{\partial x}-x\frac{\partial\tau}{\partial z}\right)$$
$$2\frac{\partial^2\xi}{\partial y\,\partial z}=f\left(z\frac{\partial\tau}{\partial z}-y\frac{\partial\tau}{\partial y}\right)\qquad 2\frac{\partial^2\xi}{\partial z\,\partial x}=2\frac{\partial^2\tau}{\partial z\,\partial t}+f\left(x\frac{\partial\tau}{\partial y}-2y\frac{\partial\tau}{\partial x}\right)\tag{2.7}$$

$$\frac{\partial^2\eta}{\partial x^2}=-fz\frac{\partial\tau}{\partial x}\qquad \frac{\partial^2\eta}{\partial y^2}=2\frac{\partial^2\tau}{\partial y\,\partial t}+f\left(z\frac{\partial\tau}{\partial x}-x\frac{\partial\tau}{\partial z}\right)$$
$$\frac{\partial^2\eta}{\partial z^2}=fx\frac{\partial\tau}{\partial z}\qquad 2\frac{\partial^2\eta}{\partial x\,\partial y}=2\frac{\partial^2\tau}{\partial x\,\partial t}+f\left(y\frac{\partial\tau}{\partial z}-2z\frac{\partial\tau}{\partial y}\right)\tag{2.8}$$
$$2\frac{\partial^2\eta}{\partial y\,\partial z}=2\frac{\partial^2\tau}{\partial z\,\partial t}+f\left(2x\frac{\partial\tau}{\partial y}-y\frac{\partial\tau}{\partial x}\right)\qquad 2\frac{\partial^2\eta}{\partial z\,\partial x}=f\left(x\frac{\partial\tau}{\partial x}-z\frac{\partial\tau}{\partial z}\right)$$

$$\frac{\partial^2\zeta}{\partial x^2}=fy\frac{\partial\tau}{\partial x}\qquad \frac{\partial^2\zeta}{\partial y^2}=-fx\frac{\partial\tau}{\partial y}$$
$$\frac{\partial^2\zeta}{\partial z^2}=2\frac{\partial^2\tau}{\partial z\,\partial t}+f\left(x\frac{\partial\tau}{\partial y}-y\frac{\partial\tau}{\partial x}\right)\qquad 2\frac{\partial^2\zeta}{\partial x\,\partial y}=f\left(y\frac{\partial\tau}{\partial y}-x\frac{\partial\tau}{\partial x}\right)\tag{2.9}$$
$$2\frac{\partial^2\zeta}{\partial y\,\partial z}=2\frac{\partial^2\tau}{\partial y\,\partial t}+f\left(z\frac{\partial\tau}{\partial x}-2x\frac{\partial\tau}{\partial z}\right)\qquad 2\frac{\partial^2\zeta}{\partial z\,\partial x}=2\frac{\partial^2\tau}{\partial x\,\partial t}+f\left(2y\frac{\partial\tau}{\partial z}-z\frac{\partial\tau}{\partial y}\right)$$

$$2\frac{\partial^2\xi}{\partial x\,\partial t}-\frac{\partial^2\tau}{\partial t^2}+f\left(y\frac{\partial\zeta}{\partial x}+y\frac{\partial\xi}{\partial z}-z\frac{\partial\xi}{\partial y}-z\frac{\partial\eta}{\partial x}\right)+g\left(3x\frac{\partial\tau}{\partial x}+y\frac{\partial\tau}{\partial y}+z\frac{\partial\tau}{\partial z}\right)=0$$
$$2\frac{\partial^2\eta}{\partial y\,\partial t}-\frac{\partial^2\tau}{\partial t^2}+f\left(z\frac{\partial\xi}{\partial y}+z\frac{\partial\eta}{\partial x}-x\frac{\partial\eta}{\partial z}-x\frac{\partial\zeta}{\partial y}\right)+g\left(x\frac{\partial\tau}{\partial x}+3y\frac{\partial\tau}{\partial y}+z\frac{\partial\tau}{\partial z}\right)=0\tag{2.10}$$
$$2\frac{\partial^2\zeta}{\partial z\,\partial t}-\frac{\partial^2\tau}{\partial t^2}+f\left(x\frac{\partial\eta}{\partial z}+x\frac{\partial\zeta}{\partial y}-y\frac{\partial\zeta}{\partial x}-y\frac{\partial\xi}{\partial z}\right)+g\left(x\frac{\partial\tau}{\partial x}+y\frac{\partial\tau}{\partial y}+3z\frac{\partial\tau}{\partial z}\right)=0$$

$$2\frac{\partial^2\xi}{\partial y\,\partial t}+f\left(y\frac{\partial\zeta}{\partial y}+z\frac{\partial\xi}{\partial x}-x\frac{\partial\xi}{\partial z}-z\frac{\partial\eta}{\partial y}-\zeta-z\frac{\partial\tau}{\partial t}\right)-\frac{zf'}{r}(x\xi+y\eta+z\zeta)+2xg\frac{\partial\tau}{\partial y}=0$$
$$2\frac{\partial^2\xi}{\partial z\,\partial t}+f\left(y\frac{\partial\zeta}{\partial z}+x\frac{\partial\xi}{\partial y}-y\frac{\partial\xi}{\partial x}-z\frac{\partial\eta}{\partial z}+\eta+y\frac{\partial\tau}{\partial t}\right)+\frac{yf'}{r}(x\xi+y\eta+z\zeta)+2xg\frac{\partial\tau}{\partial z}=0$$
$$2\frac{\partial^2\eta}{\partial x\,\partial t}+f\left(z\frac{\partial\xi}{\partial x}+y\frac{\partial\eta}{\partial z}-z\frac{\partial\eta}{\partial y}-x\frac{\partial\zeta}{\partial x}+\zeta+z\frac{\partial\tau}{\partial t}\right)+\frac{zf'}{r}(x\xi+y\eta+z\zeta)+2yg\frac{\partial\tau}{\partial x}=0$$
$$2\frac{\partial^2\eta}{\partial z\,\partial t}+f\left(z\frac{\partial\xi}{\partial z}+x\frac{\partial\eta}{\partial y}-y\frac{\partial\eta}{\partial x}-x\frac{\partial\zeta}{\partial z}-\xi-x\frac{\partial\tau}{\partial t}\right)-\frac{xf'}{r}(x\xi+y\eta+z\zeta)+2yg\frac{\partial\tau}{\partial z}=0\tag{2.11}$$
$$2\frac{\partial^2\zeta}{\partial x\,\partial t}+f\left(x\frac{\partial\eta}{\partial x}+y\frac{\partial\zeta}{\partial z}-z\frac{\partial\zeta}{\partial y}-y\frac{\partial\xi}{\partial x}-\eta-y\frac{\partial\tau}{\partial t}\right)-\frac{yf'}{r}(x\xi+y\eta+z\zeta)+2zg\frac{\partial\tau}{\partial x}=0$$
$$2\frac{\partial^2\zeta}{\partial y\,\partial t}+f\left(x\frac{\partial\eta}{\partial y}+z\frac{\partial\zeta}{\partial x}-x\frac{\partial\zeta}{\partial z}-y\frac{\partial\xi}{\partial y}+\xi+x\frac{\partial\tau}{\partial t}\right)+\frac{xf'}{r}(x\xi+y\eta+z\zeta)+2zg\frac{\partial\tau}{\partial y}=0$$

$$\frac{\partial^2\xi}{\partial t^2}+f\left(y\frac{\partial\zeta}{\partial t}-z\frac{\partial\eta}{\partial t}\right)+g\left(\xi+2x\frac{\partial\tau}{\partial t}-x\frac{\partial\xi}{\partial x}-y\frac{\partial\xi}{\partial y}-z\frac{\partial\xi}{\partial z}\right)+\frac{xg'}{r}(x\xi+y\eta+z\zeta)=0$$

$$\frac{\partial^2\eta}{\partial t^2}+f\left(z\frac{\partial\xi}{\partial t}-x\frac{\partial\zeta}{\partial t}\right)+g\left(\eta+2y\frac{\partial\tau}{\partial t}-x\frac{\partial\eta}{\partial x}-y\frac{\partial\eta}{\partial y}-z\frac{\partial\eta}{\partial z}\right)+\frac{yg'}{r}(x\xi+y\eta+z\zeta)=0 \tag{2.12}$$

$$\frac{\partial^2\zeta}{\partial t^2}+f\left(x\frac{\partial\eta}{\partial t}-y\frac{\partial\xi}{\partial t}\right)+g\left(\zeta+2z\frac{\partial\tau}{\partial t}-x\frac{\partial\zeta}{\partial x}-y\frac{\partial\zeta}{\partial y}-z\frac{\partial\zeta}{\partial z}\right)+\frac{zg'}{r}(x\xi+y\eta+z\zeta)=0$$

where ′ denotes differentiation with respect to r.

The solution of equations (2.6)-(2.12) was performed block by block. Although the calculations are lengthy, they are routine and we merely quote the results. For general f and g the equation

$$\ddot{\boldsymbol{r}}+f(r)\boldsymbol{L}+g(r)\boldsymbol{r}=0 \tag{2.13}$$

obviously has the algebra $a_1 \oplus so(3)$ with a_1 representing invariance under time translation and so(3) the usual rotational invariance. Specifically, the generators of so(3) are

$$G_1=z\frac{\partial}{\partial y}-y\frac{\partial}{\partial z} \qquad G_2=x\frac{\partial}{\partial z}-z\frac{\partial}{\partial x} \qquad G_3=y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y} \tag{2.14}$$

and the generator of a_1 is

$$G_4=\frac{\partial}{\partial t}. \tag{2.15}$$

Special cases of (2.13) with additional symmetry are

$$\ddot{\boldsymbol{r}}+\frac{\mu\boldsymbol{r}}{r^{4A/(K+A)}}=\boldsymbol{0} \tag{2.16}$$

which has the algebra $a_2 \oplus so(3)$, the additional symmetry being

$$G_5=2At\frac{\partial}{\partial t}+(A+K)\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right) \tag{2.17}$$

$$\ddot{\boldsymbol{r}}+\frac{\lambda\boldsymbol{L}}{r^3}+\frac{\mu\boldsymbol{r}}{r^4}=\boldsymbol{0} \tag{2.18}$$

which has the algebra $sl(2, R) \oplus so(3)$ with

$$G_6=t^2\frac{\partial}{\partial t}+t\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right) \tag{2.19}$$

and

$$\ddot{\boldsymbol{r}}+\frac{\lambda\boldsymbol{L}}{r^3}+\left(\frac{\mu}{r^4}-\varepsilon\right)\boldsymbol{r}=\boldsymbol{0} \tag{2.20}$$

which has the same algebra, but now

$$G_5=e^{2t\sqrt{\varepsilon}}\left(\frac{1}{\sqrt{\varepsilon}}\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right) \tag{2.21}$$

$$G_6=e^{-2t\sqrt{\varepsilon}}\left(\frac{1}{\sqrt{\varepsilon}}\frac{\partial}{\partial t}-x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}-z\frac{\partial}{\partial z}\right). \tag{2.22}$$

(For $\varepsilon<0$, G_5 and G_6 can be replaced by $\bar{G}_5$ and $\bar{G}_6$ which would be expressed in terms of sine and cosine functions.)

The additional symmetry (2.17) for the power-law central force represents invariance under the self-similar transformation

$$t = \alpha^{2A}\bar{t} \qquad \boldsymbol{r} = \alpha^{K+A}\bar{\boldsymbol{r}} \tag{2.23}$$

and is lost if the central force is not a power law.

The equations of motion (2.18) and (2.20) both possess the algebra sl(2, R)$\oplus$so(3) regardless of the values of the parameters. They may be regarded as direct extensions of the results of Moreira *et al* (1985) for the algebra of the charge monopole problem.

The term $\mu \boldsymbol{r} r^{-4}$ can be interpreted as a centripetal force and the term $\varepsilon \boldsymbol{r}$ represents a harmonic repulsor or oscillator depending upon the sign of ε or a 'free' particle if $\varepsilon = 0$. The latter term does not affect the algebra nor the integrability of the equation. Indeed the same conserved vector exists as for (1.5), namely

$$\boldsymbol{P} = \boldsymbol{L} - \frac{\lambda \boldsymbol{r}}{r} \tag{2.24}$$

and L is also constant in both cases. We note that the motion continues to be on the surface of a cone. Another scalar integral

$$I = \tfrac{1}{2}(\dot{\boldsymbol{r}}^2 - \mu/r^2 - \varepsilon \boldsymbol{r}^2) \tag{2.25}$$

also exists. Equation (2.20) does not belong to the class of problems (1.13) treated by Leach and Gorringe (1988). However, (2.18) does in the particular case that $\mu = -\lambda^2$. Then, in addition to the three integrals above, there is also the conserved vector

$$\boldsymbol{J} = \boldsymbol{L} \times \dot{\boldsymbol{r}} - \frac{\lambda \boldsymbol{L}}{r} = \boldsymbol{P} \times \dot{\boldsymbol{r}} \tag{2.26}$$

from which the orbit equation is just as easily obtained by taking the scalar product with $\boldsymbol{r}$ as was the case with (2.24).

3. Connection between Laplace–Runge–Lenz-like vectors and symmetries

We have seen that, when in (2.13) $f(r) \equiv 0$ and $g(r)$ is a power law, the algebra of the symmetries is $a_2 \oplus$so(3) no matter the degree of the power law. However, when $g(r)$ is not a power law, G_5 is lost. Now, in the case of the Kepler problem G_5 is usually associated with the existence of the Laplace-Runge-Lenz vector of that problem (Lévy-Leblond 1971). However, Leach (1981) demonstrated that the vector could be obtained from G_4, which represents invariance under time translation, a property of any autonomous first integral. As Fradkin (1965) has demonstrated, any central force law will have a Laplace-Runge-Lenz-like vector and so it is evident that the presence of G_5 is not essential for its existence although it may well help in the determination of an explicit expression for the vector.

In this paper, because we were interested in equations of the types (1.1), (1.5) and (1.13), we have not considered the general planar problem for which Yoshida (1989) demonstrated the existence of a Laplace-Runge-Lenz-like vector. However, elsewhere (Gorringe and Leach 1987) we showed that an explicit expression for a Laplace-Runge-Lenz-like vector for the autonomous equation of motion

$$\ddot{\boldsymbol{r}} + g\hat{\boldsymbol{r}} + h\hat{\boldsymbol{\theta}} = \boldsymbol{0} \tag{3.1}$$

could be found by Collinson's method (Collinson 1973) only when

$$g(r, \theta) = \frac{U''(\theta) + U(\theta)}{r^2} + \frac{2V'(\theta)}{r^{3/2}} \tag{3.2}$$

$$h(r, \theta) = \frac{V(\theta)}{r^{3/2}} \tag{3.3}$$

where U and V are arbitrary functions of θ. Excluding the cases for which the force is constant or $h \equiv 0$ and g a power-law potential, one finds that (3.1) possesses one symmetry in addition to the obvious one of the generator of time translations only when

$$g = (K \sin 2\theta + L \cos 2\theta) W(w_1) G + X(w_1) G \tag{3.4}$$

$$h(r, \theta) = G^{-1} W(w_1) \tag{3.5}$$

where

$$G = (F + K \cos 2\theta - L \sin 2\theta)^{-1/2} \exp\left((E - 2A_1) \int \frac{d\theta}{F + K \cos 2\theta - L \sin 2\theta} \right) \tag{3.6}$$

$$w_1 = r^2 (F + K \cos 2\theta - L \sin 2\theta) \exp\left(-2E \int \frac{d\theta}{F + K \cos 2\theta - L \sin 2\theta} \right) \tag{3.7}$$

and W and X are arbitrary functions of w_1 and A_1, F, K and L are arbitrary constants. Even when the r dependence in (3.4) is put in the form of (3.2) and (3.3), which is possible, the freedom of choice of $U(\theta)$ and $V(\theta)$ is restricted. (The method of calculation follows that outlined in section 2.)

The point which may be inferred from the two previous paragraphs is that the number of symmetries for a planar motion is not related to the existence of a conserved vector of Laplace-Runge-Lenz type at all. However, in terms of the construction of an explicit expression for the vector, the existence of symmetries provides a method. A function $I(r, \theta, \dot{r}, \dot{\theta}, t)$ is invariant under the action of a symmetry G if $G^{[1]} I = 0$. In principle we can find I as a function of four independent characteristics. The requirement that I be a first integral, i.e. $\dot{I} = 0$, reduces this number to three independent characteristics each of which is a first integral. It is in this sense that the existence of symmetries is of value. It is not in the demonstration of existence but in the process of construction of explicit expressions for the first integrals.

Finally we remark that one of the main results of this paper has been the determination of the Lie algebras of the Lie point symmetries admitted by (1.19). We have seen that the Kepler problem shares the same algebra with any power-law potential. Moving away from planar motions, the algebra of the symmetries of (2.18) and (2.19) is independent of the values of the parameters λ, μ and ε (excepting that λ and μ may not both be zero). By way of contrast, McIntosh and Cisneros (1970) in their study of the monopole-Kepler and monopole-oscillator problems take $\mu = -\lambda^2$ to ensure closed orbits. This choice had already been made for the former problem by Zwanziger (1968) so that a Laplace-Runge-Lenz-like vector could be explicitly constructed. (The reason for the case of construction of the vector becomes obvious from the more general work on the construction of Laplace-Runge-Lenz-like vectors for (1.19) by Leach and Gorringe (1988).) It would be interesting to see whether the relaxation of this condition, which is not required for the symmetries of the differential equation (the source of a construction base), leads to significant results for the monopole-oscillator problem.

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